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INTERPOLATING A-PARCHES CONSTRUCTED WITH CUBOIDS. INTERPOLATING A-PATCHES BUILT WITH CUBOIDS.

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Summary

The objective of this research is to design and control a segment of a degree three algebraic surface or cuboid. The cuboid is required to interpolate two given conics, which lie in two different planes in space, this is called an interpolation A-Patch. To achieve this objective, the expression of the cuboid in tetrahedral coordinates was studied and the property that these surfaces contain straight lines was used. The conditions for connecting two segments of cuboids with class G^1 This means, coincidence or continuity of the tangent planes on the contact curve of the cuboid segments. The connection of two or more cuboid segments with class G^1 is what is called an A-Patch. It is also emphasized that a certain family of cuboid planar profiles are ellipses for the construction of tubular surfaces.

Keywords: Cuboid, class connection, A-Parches, tetrahedral coordinates, conics. G^1 A-Parches, tetrahedral coordinates, conics.

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CONTENIDO

Introduction

The background to this research are the studies of certain tubular surfaces designed with the envelope of a monoparametric quadratic family of spheres presented in: Bohem, W. and Paluszny, M. (1998), Paluszny, M. and Boehm, W. (1998) and Franquiz J.; Paluszny, M. and Tovar, F. (2006). The implicit equation of such tubular surfaces has in general degree 4, and in certain particular cases it is of degree 3, that is, the tubular surface is a cuboid. Since these tubular surfaces consist of circular profiles, it is possible to prescribe two interpolation circles for the design of tubular splines. These tubular splines are generally constructed with class G^1 . Our study generalizes the expression of this tubular surface of degree 3, so that it interpolates any pair of conics lying in two different planes in space, using a section of this surface. In particular, emphasis is placed on the case that the given interpolation conics are ellipses and that certain plane profiles of the cuboid are also elliptic. In this way, tubular surfaces of degree three with elliptic profiles can be designed. In Xu G.; Huang, H. and Bajaj, C. (2001), splines are constructed with segments of algebraic surfaces, which are called A-Parches, in our case we are going to use specifically algebraic surfaces of degree 3 to construct the A-Parches. In this construction we use cuboid segments in tetrahedral coordinates and we study the connection of two surface segments on the cuboid trace with the tetrahedron. The construction of an interpolating cuboid in tetrahedral coordinates is proposed by controlling each surface profile. To do this, the family of cuboids is considered such that one of its straight lines coincides with one of the edges of the tetrahedron. In this way the conical profiles of the cuboid are the consequence of ablating the surface from the initial conic of interpolation to the final conic, with a family of planes that have that straight line in common. The process is similar to the case of tubular surfaces described in Bohem, W. and Paluszny, M. (1998), Franquiz, J.; Paluszny, M. and Tovar, F. (2006). The segments constructed in this article interpolate a sequence of conics given in space and can be connected with class G^1.

Note that if one considers the problem of interpolating two conics in R^3 which lie in two different planes, with an algebraic surface of degree two or quadric, one does not have enough degrees of freedom to solve the problem. For this reason cuboids are used.

2. Algebraic surface of degree three or cuboid

The general expression of a Cuboid in tetrahedral coordinates (s, t, u, v) is given by:

$$C(s,t,u,v)=\sum_{i=1}^{20} a_i s^l t^m u^n v^k, l+m+n+k=3, a_i \in \mathbb{R} \quad (1)$$

If the surface $C(s,t,u,v)=0$ contains the line $s=0, u=0$ its expression reduces to:

$$C_a(s,t,u,v)=uCs_a(t,u,v)+sCu_a(s,t,v)+suP_a(s,t,u,v) \quad (2)$$

Where:

$$Cs_a(t,u,v)=a_1u^2+\ a_2tu+a_3t^2+a_4uv+a_5v^2+a_6tv \quad (3)$$

$$Cu_a(s,t,v)=a_7s^2+a_8ts+a_9t^2+a_{10}sv+a_{11}v^2+a_{12}tv \quad (4)$$

$$P_a(s,t,u,v)=a_{13}s+a_{14}t+a_{15}u+a_{16}v \quad (5)$$

The subscript a is necessary to distinguish two different cuboids when studying the connection between the G^1.

Intersecting the cuboid given by (2) with the plane $s=0$ results in the reducible cubic $uCs_a(t,u,v)=0$ and similarly the reducible cubic results $sCu_a(s,t,v)\ =0$ when intersecting the equation (2) with the plane $u=0$.

The coefficients defining the plane $P_a(s,t,u,v)=0$ are free and will be used to control the type of cuboid profile and the connection between two surface segments. The cuboid interpolates the conics preset by $Cs_a(t,u,v)=0$ in the plane $s=0$ y $Cu_a(s,t,v)=0$ in the plane $u=0$. An example of a cuboid containing a straight line is shown in Figure 1.

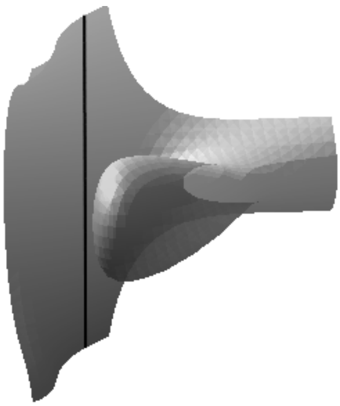
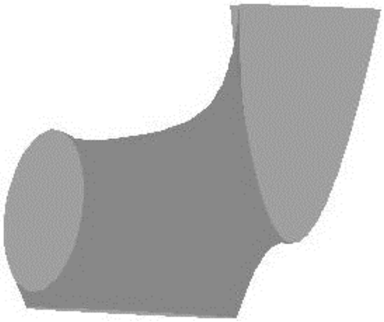


Figure 1. Example of a cuboid containing a straight line.

Note that the initial and final interpolation conics of the cuboid are prescribed, with the coefficients of conics (3) y (4). Here it is evident that if one intends to do this procedure using surfaces of degree 2The cuboid is not free enough parameters to define the two interpolation conics independently. Figure 2 shows a cuboid segment interpolating two given conics.



Cuboid interpolating two given conics, in this case an ellipse and a parabola.

3. Connection of two class cuboid segments G^1.

Suppose we have two cuboids, the first given by the equation (2) and the second given by:

$$C_b(s,t,u,v) = uCs_b(t,u,v) + sCu_b(s,t,v) + suP_b(s,t,u,v) \tag{6}$$

Where:

$$Cs_b(t,u,v) = \alpha a_1 u^2 + \alpha a_2 tu + \alpha a_3 t^2 + \alpha a_4 uv + \alpha a_5 v^2 + \alpha a_6 tv \tag{7}$$

$$Cu_b(s,t,v) = b_7 s^2 + b_8 ts + b_9 t^2 + b_{10} sv + b_{11} v^2 + b_{12} tv \tag{8}$$

$$P_b(s,t,u,v) = b_{13} s + b_{14} t + b_{15} u + b_{16} v \tag{9}$$

The equation (6) shows a second cuboid that connects with the first cuboid with class G^0 since they coincide on the plane $s = 0$. Figure 3 shows two connected cuboids with class G^0.

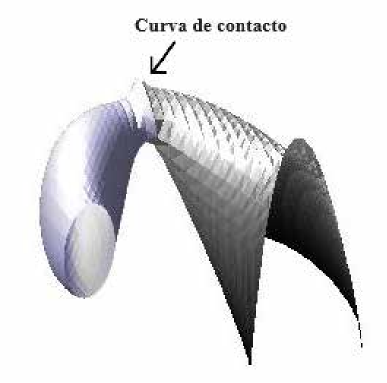


Figure 3. Example of two class-connected cuboid segments. G^0.

By establishing the conditions for the family of tangent planes to coincide along the common conic in the plane $s = 0$we obtain the coefficients of the second cuboid given by (6):

$$\{b_9 = \alpha a_9, b_{11} = \alpha a_{11}, b_{12} = \alpha a_{12}, b_{14} = \alpha a_{14}, b_{15} = \alpha a_{15}, b_{16} = \alpha a_{16}\}$$

where α is a real parameter.

This is the solution of the system of equations, which results from calculating the coefficients of the family of tangent planes, on the interpolation curve lying on the plane $s = 0$ and find the conditions for all of them to coincide on this conic. These conditions subtract degrees of freedom to the choice of the conic of interpolation of the second cuboid on the plane $u = 0$ given by the expression (6) leaving only three free coefficients for the design of this conic, these are: b_7, b_8 y b_{10} from equation (8). The coefficient b_{13} of the expression (9) is free and allows to dilate or contract the cuboid segment $C_b(s,t,u,v)$.

In conclusion, for two cuboids to be connected with class G^1 the equation of the first must be given by (2) and the equation of the second must be given by (6), where:

$$Cs_b(t,u,v) = \alpha a_1 u^2 + \alpha a_2 tu + \alpha a_3 t^2 + \alpha a_4 uv + \alpha a_5 v^2 + \alpha a_6 tv \qquad (10)$$

$$Cu_b(s,t,v) = b_7 s^2 + b_8 ts + \alpha a_9 t^2 + b_{10} sv + \alpha a_{11} v^2 + \alpha a_{12} tv \qquad (11)$$

$$P_b(s,t,u,v) = b_{13} s + \alpha a_{14} t + \alpha a_{15} u + \alpha a_{16} v \qquad (12)$$

In Figure 4, two cuboids can be observed to be connected with class G^1 since the tangent planes coincide along the contact curve. It interpolates an initial ellipse, connects into another ellipse and ends in a parabola.

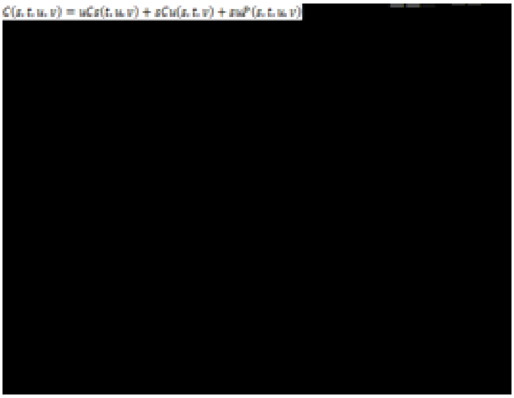


Figure 4. Two class-connected cuboids. G^1.

By manipulating the free coefficients in the second cuboid, other interpolating A-Parches can be obtained. Figure 5 shows an example of two cuboids that are connected by class G^1 generating two elliptic tubes.

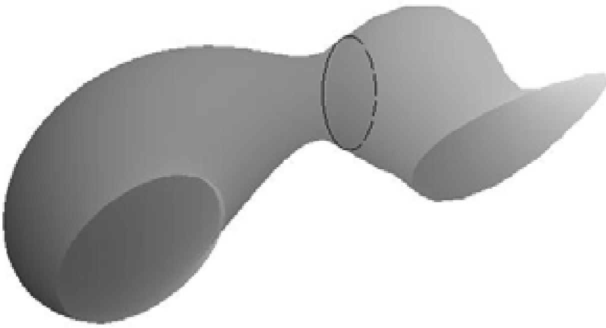


Figure 5. Two cuboids formed by elliptical profiles, connected with class G^1.

4. Cuboid with elliptical profiles

In the case of the construction of classically studied tubular surfaces, such as cichlids, their profiles are circles (see Paluszny, M. and Boehm, W. (1998), Paluszny, M. and Tovar, F. (2006)). For a cuboid given by (2) the surface profiles are conic. In particular, for the construction of tubular surfaces using cuboids, we will use elliptic profiles. The complete analysis of this case is published in Tovar, F., Otero, J. and Daza, J. (2023). However, the most relevant facts are presented:

So that the cuboid given by (note that the following expression coincides with the one given in (2), only that it does not have the subscript *a*):

$$C(s,t,u,v) = uCs(t,u,v) + sCu(s,t,v) + suP(s,t,u,v)$$

is made up of elliptical profiles, two geometrical conditions are required:

1) The cubic at infinity of the cuboid must be reducible, that is, the product of a conic with a straight line.

2) The conic defined by the cubic at infinity must be imaginary.

These two properties guarantee that the conics forming the cuboid profiles are ellipses.

If the cubic at infinity is given by the product of a straight line with a conic, the expression of this curve must be of the form:

$$C_i(s,t,u) = (As^2 + Bsu + Cu^2 + Dst + Etu + Ft^2)(Hs + Jt + Ku) \tag{13}$$

where the coefficients of the expression (13) depend on the interpolation conics and the plane given in (5)that is, the coefficients a_i of the cuboid. Two cases are derived, the first one $F \neq 0$ y $J = 0$. The coefficients of the conic at infinity are determined by the expression:

$$H = \frac{a_{11} - a_{12} + a_9}{F} \qquad K = \frac{a_3 + a_5 - a_6}{F}$$

$$A = \frac{a_{10} - a_{11} - a_7}{H} \qquad C = \frac{a_1 - a_4 + a_5}{K}$$

$$D = \frac{-a_{10} + 2a_{11} - a_{12} + a_8}{H}$$

$$E = \frac{a_2 - a_4 + 2a_5 - a_6}{K}$$

$$a_{13} = AK + BH + a_{10} - 2a_{11} + a_{16} - a_5$$

$$a_{14} = DK + EH - 2a_{11} + a_{12} + a_{16} - 2a_5 + a_6$$

$$a_{15} = BK + CH - a_{11} + a_{16} + a_4 - 2a_5$$

Where B, F y a_{16} are free parameters used to modify the shape of the interpolation cuboid. The cubic at infinity is defined in terms of the two interpolation conics and the parameters B, F y a_{16}.

Finally, the expression of the interpolation cuboid obtained is:

$$C(s,t,u,v) = uCs(t,u,v) + sCu(s,t,v) + suP(s,t,u,v) \tag{14}$$

where:

$$P(s,t,u,v) = (AK + BH + a_{10} - 2a_{11} + a_{16} - a_5)s + (DK + EH - 2a_{11} + a_{12} + a_{16} - 2a_5 + a_6)t + (BK + CH - a_{11} + a_{16} + a_4 - 2a_5)u + a_{16}v$$

Using the fact that the prescribed interpolation conics are ellipses, it is guaranteed that the cuboid profiles are also ellipses, for certain values of B. The polynomial $P(B)$ has real roots and it is sufficient to choose the value of B between both roots if $F > 0$ and in the case $F < 0$ is chosen B out of the roots of the polynomial.

The second case: $F = 0$ y $J \neq 0$ the equation of the cubic at infinity is expressed:

$C_i(s,t,u) = (As^2 + Bsu + Cu^2 + Dst + Etu)(Hs + Jt + Ku)$. Similar to the previous case when substituting $v = -s - t - u$ in equation (2) in the previous case, the result is nonlinear equations for the coefficients A, B, C, D, E, H, J & K These are:

$$A = -\frac{a_{10} - a_{11} - a_7}{H} \qquad C = \frac{a_1 - a_4 + a_5}{K}$$

$$D = \frac{a_{11} - a_{12} + a_9}{J} \qquad E = \frac{a_3 + a_5 - a_6}{J}$$

$$a_{13} = AK + BH + a_{10} - 2a_{11} + a_{16} - a_5$$

$$a_{14} = BJ + DK + EH - 2a_{11} + a_{12} + a_{16} - 2a_5 + a_6$$

$$a_{15} = BK + CH - a_{11} + a_{16} + a_4 - 2a_5$$

$$H = -\frac{AJ + a_{10} - 2a_{11} + a_{12} - a_8}{D} \qquad K = -\frac{CJ - a_2 + a_4 - 2a_5 + a_6}{E}$$

The expressions of the coefficients A y H are related. For both equations to be satisfied it is necessary to calculate the zeros of a polynomial of degree two in $[H,J]$ given by :

$$P[H,J] = (a_{12} - a_{11} - a_9)H^2 - (a_{10} - 2a_{11} + a_{12} - a_8)HJ + (a_{10} - a_{11} - a_7)J^2$$

The discriminant of the polynomial $P[H,J]$ denoted by $\Delta(C_u)$ which is positive because the conic is an ellipse. C_u an ellipse. Therefore, it has a solution and its two roots can be calculated.

A similar fact occurs with the expressions for the coefficients of C y K and the polynomial results from both expressions:

$$P[K,J] = (-a_3 - a_5 + a_6)K^2 - (a_2 - a_4 + 2a_5 + a_6)KJ - (a_1 - a_4 + a_5)J^2$$

The determinant of the polynomial $P[K,J]$ denoted by $\Delta(C_s)$ is positive because the conic C_s an ellipse, therefore, it has a solution and its two roots can be calculated.

By calculating the values for H y K from the polynomials $P[H,J]$ y $P[K,J]$the rest of the coefficients can be calculated, leaving free the coefficients B,J y a_{16}.

These elliptic profiles can be real or imaginary. In the case of imaginary ellipses, a disconnection of the cuboid appears, this is controlled by the free coefficients of the cuboid. For the cuboid given by (14) the free parameters allow to control the shape of the surface.

Figure 6 shows a cuboid that interpolates two ellipses and in addition all the profiles that are produced by ablating with the planes are also elliptic.

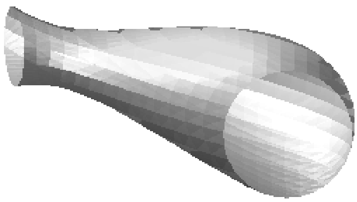


Figure 6. Cuboid formed by elliptical profiles.

5. Conclusion

From the analysis performed on algebraic surfaces of degree 3, it is concluded that it is possible to construct and control segments of algebraic surfaces that interpolate two conics in space. In the analysis of these cuboids, the property that these algebraic surfaces have of containing straight lines was used. In particular, in the case that the cuboid contains only one straight line, this straight line was used to fan a family of planes, which when intercepting the cuboid, results in a family of conics in $3D$ similar to the tubular surfaces described in Paluszny, M. Boehm, W. (1998), called cichlides. With this method, an algebraic surface segment is constructed whose profiles are conics and interpolates two prescribed conics, see Figure 1. G^0 see Figure 3, i.e., interpolation along a conic in space and of class G^1 which means greater smoothness on the contact curve, since, on this curve, the tangent planes of both cuboids coincide, see Figures 4 and 5. The resulting free coefficients for the control of the cuboids, allow to modify their shape while preserving the interpolation conditions. Figure 7 shows the change of the cuboid segment when varying the parameter a_{16}.

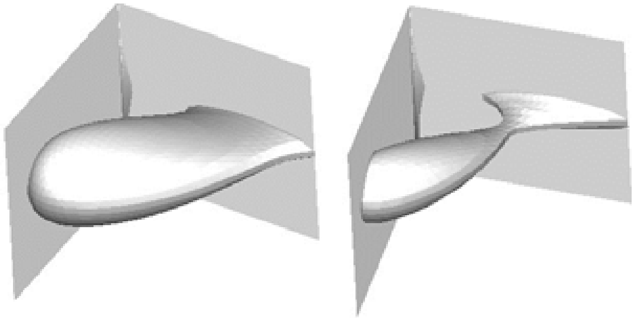


Figure 7. Change of the cuboid shape when modifying the parameter. a_{16}.

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